Chapter 3

Deriving Weights from Pairwise Comparison Matrices by Goal Programming

3.1 Introduction

There are four basic steps in AHP: construction of hierarchy which includes salient objects (criteria, alternatives, etc.) of the problem, formation of pairwise comparison matrices, elicitation of the local priority weights from the comparison matrices, and synthesis of these local priorities to obtain global (or overall) weights for the alternatives. The present chapter is concerned with the important third step, i.e., how to elicit weights from pairwise comparison matrices.

The issue of deriving weights from pairwise comparison matrices has attracted considerable attention in recent years and resulted in the development of several scaling methods. But unfortunately, there has not been any consensus about the choice of method to determine weights. In fact, the methodology should be such that the weights derived by it 'best' fit the data represented by the matrix **A** in (2.1). Different methods take this 'best fitting' in different ways. For completeness of this chapter, we repeat some discussions on weight determination methods.

Saaty (1977a), in his pioneering work, suggested the eigenvector method (EM). The superiority of EM over other methods is also discussed by Saaty and Vargas (1984b) and Saaty (1990c). Crawford and Williams (1985) and Crawford (1987) have advocated the logarithm least squares method (LLSM) which is also known as geometric mean method. They have discussed the preferability of LLSM to the EM with respect to several properties of ratio-scale matrices. Barzilai et al. (1987) algebraically validated the works of Crawford and Williams (1985) on LLSM. Another way of weight determination, the least squares method (LSM), is strongly suggested by Jensen (1984, 1989). In a recent paper, Bryson (1995) has used logarithmic goal programming method to derive the underlying weights from PCMs. Apart from these methods, some other methods are also available in AHP literature. Few of these are: row-column averaging procedure (Saaty, 1977a), Chi-square method (Jensen, 1988), logarithmic absolute deviation method (Cook and Kress, 1988), a variant of eigenvector method (Takeda et al., 1987). A few more papers (e.g., Zahedi, 1986a; Golany and Kress, 1993) are mainly concerned with the question of comparison of several methods based upon their performances on various criteria.

3.2 No Consensus on the Choice of Methodology and Proposition of a New Technique

It is worth mentioning that all the methods mentioned in the previous section give exactly the same set of weights when the comparison matrix is consistent, i.e., it satisfies the consistency relation $a_{ij} = a_{ik}a_{kj}$ for all i, j, and k. But people are inherently inconsistent in their own judgments, especially if they are dealing with fuzzy concepts, such as quality, attractiveness, comfort, etc. Where there is inconsistency, various methods give diverse results and this diversity is proportional to the amount of inconsistency captured in the entries of comparison matrices. Obviously, there should be a certain limit of inconsistency, beyond which the solution might not be acceptable. One of the mentionable advantages of Saaty's (1977a) eigenvector method (EM) is that it gives an index of inconsistency. According to him, elicited weights will be acceptable if consistency ratio (C.R.) of the matrix is lees than 0.10. Further, the eigenvector method is primarily concerned with preserving ordinal priorities. But EM fails to preserve "inverse reciprocity" property (described in the previous chapter), whereas LLSM preserves it. This is a remarkable advantage of LLSM over EM. On the other hand, least squares method (LSM) is invoked to obtain the weights so as to minimize the deviational error

$$\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - w_i / w_j)^2$$

With respect to the foregoing criterion of minimization of Euclidean distance, LSM outperforms both EM and LLSM. But computationally, LSM is not at all favorable. It is also to be noted that neither LLSM nor LSM gives any well-accepted index measuring inconsistency, however, they satisfy the important ratio-scale matrix property, 'inverse reciprocity'. Although, Bryson's (1995) goal programming method has some desirable properties, but in this method, instead of dealing with the actual responses provided by the DM, the author has dealt with their corresponding logarithmic values; this necessitates subsequent transformation and normalization of the computed weights. It is now clear that each method has some advantages and some disadvantages. But every method gives optimal solution with respect to the criterion for which it is designed. Golany and Kress (1993, page 219) writes:

"The choice of method should be dictated by the objectives of the analysis and the desired measure of effectiveness. Evidently, different objectives may result in different scaling methods."

It is already mentioned that LSM is optimal under the criterion 'minimization of Euclidean distances' between actual responses, i.e., the a_{ij}'s, and their corresponding consistency adjusted surrogates, namely, $u_{ij} = w_i / w_j$'s. In this chapter, we are concerned with the 'minimization of absolute deviation between a_{ij}'s and u_{ij}'s (i.e., maximization of the number of satisfied cells in the matrix). We assume that, while obtaining estimates of weights from comparison matrices, decision maker's (DM) goal is to get the weights whose pairwise ratios are closest to a_{ij}'s. This motivation leads to the problem of minimization of absolute deviations, not the sum of squares of errors (SSE), as it is done in LSM.

In the next section, we show how the goal programming technique can be used to minimize the absolute deviations between a_{ij}'s and u_{ij}'s.

3.3 Goal Programming as a Weight Determination Technique

To obtain the absolute deviation ratio-scale weights w_i, i = 1, 2,…, n, from the ratio-scale comparison matrix **A** in (2.1), we

$$\text{Minimize:} \quad \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}\left|a_{ij} - u_{ij}\right|$$

$$\text{subject to} \quad u_{ij} = w_i / w_j, \qquad \sum_{i=1}^{n} w_i = 1, \quad w_i > 0$$

The foregoing weight derivation minimization problem can be reformulated as a lexicographic goal programming problem:

$$\text{Minimize lexicographically:} \quad \text{a} = \left(n_{nn} + p_{nn},\ \sum_{i=1}^{n-1}\sum_{j=i+1}^{n}(n_{ij} + p_{ij})\right)$$

$$\text{subject to} \quad w_i - a_{ij} w_j + n_{ij} - p_{ij} = 0, \quad i = 1, 2, ..., n-1, \quad j = i+1, ..., n$$

$$\sum_{i=1}^{n} w_i + n_{nn} - p_{nn} = 1$$

$$w_i > 0, \ i = 1, 2, ..., n$$

It is to be noted that satisfaction of the normalization constraint and the sum of all other deviational variables, n_{ij} *and* p_{ij}, i = 1, 2,…, n-1; j = i + 1, …, n have been kept at the first and second priority levels, respectively.

Actually, there is a number of least distance approximation models for finding estimates of **w** in the sense of minimizing the distances of the response elements a_{ij} from their corresponding consistency adjusted surrogates u_{ij}. Although the distance measure in the least squares method is a reasonable and widely used measure of deviation, but this is not the case in the weight extraction problem. Here, the decision analyst is interested to obtain the weights whose ratios are closest to a_{ij} values. This necessitates the use of absolute deviation minimization technique. Typically, this job can be done by goal programming which has a number of advantages over the least squares method.

3.4 Advantages of Goal Programming Method (GPM)

Since we are concerned with the problem of minimization of distances between a_{ij}'s and u_{ij}'s, we compare goal programming method with LSM and LLSM. Following are some of the advantages of the goal programming method over LSM and LLSM:

1. There is no well-established algorithm to derive weights from ratio-scale matrices by using the LSM. The algorithm of LSM mentioned in Saaty and Vargas (1984b) is not computationally efficient. This is the major disadvantage of using LSM, whereas, using goal programming we can easily calculate the weights. Many computer codes are also available for goal programming.

2. In general, goal programming method (GPM) satisfies the maximum number of elements in the comparison matrix (verified empirically). This is a remarkable advantage of GPM over other methods (see Table 3.4).

3. There may be no comparison or more than one comparisons for some cells in the comparison matrices. In such cases, LLSM minimizes

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n_{ij}}[\ln a_{ij} - (\ln w_i - \ln w_j]^2$$

where n_{ij} may be zero (no judgment is available) or greater than one (indicating multiple judgments). Note that LLSM does not yield any closed form solution (Crawford, 1987) in this case. Although a problem of multiple decision makers can be tackled by LSM, but it is even more difficult than the case of single DM. Evidently, the case of no judgment or multiple judgments can be effectively tackled by GPM. For multiple decision makers, the set of all the pertinent judgments for a specific comparison may be taken as a closed interval.

4. While forming comparison matrices, DM may not be equally certain about all the pairs of comparisons, i.e., the degree of confidence for all the comparisons may not be equal (by degree of confidence, we mean how easily the DM has specified his judgments). Further, it can be easily checked that the marginal impacts of all the cells on the resulting solution are not equal (Takeda et al., 1987). In addition to this, DM may prefer to satisfy some particular cells for some reason. If this is the case, then it can be easily accomplished by placing the concerned constraints at the higher priority level in the goal programming formulation. This task cannot be accomplished by LSM or LLSM.

5. In course of articulating preferences, DM may give his/her strength of preference as ‘at least 3’ or ‘at most 5’ for example. This type of preferences can be considered by minimizing the appropriate deviational variable in the achievement vector **a**. It is to be noted that LSM and LLSM lack this advantage of goal programming.

6. The task of sensitivity analysis can easily be performed by goal programming method (see Ignizio, 1982, Chapter 19).

3.5 Numerical Example

In this section, we extract weights from the following two comparison matrices adopted from Saaty and Vargas (1984b). In the first matrix, the departure from consistency is relatively large, whereas in the second one, the departure from consistency is not that much.

Example 3.1:

	O_1	O_2	O_3	O_4
O_1	1	4	½	1/5
O_2	¼	1	1/3	4
O_3	2	3	1	½
O_4	5	¼	2	1

Example 3.2:

	O_1	O_2	O_3	O_4	O_5
O_1	1	1/6	1/3	1/8	5
O_2	6	1	2	1	8
O_3	3	½	1	½	5
O_4	8	1	2	1	5
O_5	1/5	1/8	1/5	1/5	1

The consistency ratio (C.R.) of the matrix in example 3.1 is 0.896, which is much greater than 0.10, but the matrix of example 3.2 is not that much inconsistent. Its C.R. = 0.079, which is less than 0.10. The relative weights determined by LLSM, LSM, and GPM for all the objects in Examples 3.1 and 3.2 are shown in Table 3.1 and Table 3.2, respectively. The total absolute deviation (TAD) and error sum of squares (SSE) are also shown in these tables. The individual absolute deviations for all the three methods are shown in Table 3.3 (for Example 3.1) and Table 3.4 (for Example 3.2).

Table 3.1: Normalized weights of the four objects (Example 3.1)

Methods	w_1	w_2	w_3	w_4	TAD	SSE
LLSM	0.1930	0.1840	0.3190	0.3040	15.8687	37.8049
LSM	0.1180	0.1750	0.2930	0.4080	14.8840	34.5373
GPM	0.1304	0.0870	0.2609	0.5217	13.5504	55.1473

Table 3.2: Normalized weights of the five objects (Example 3.2)

Method	w_1	w_2	w_3	w_4	w_5	TAD	SSE
LLSM	0.0730	0.3580	0.1870	0.0360	0.0360	15.4836	46.0796
LSM	0.0550	0.3630	0.2010	0.3320	0.0490	12.1545	24.6653
GPM	0.0597	0.3582	0.1791	0.3582	0.0448	10.3876	27.7405

Table 3.3: Individual absolute deviations

$|a_{ij} - u_{ij}|$ (Example 3.1)

Method	Col. 1	Col. 2	Col. 3	Col. 4
LLSM	0.0000	2.9511	0.1050	0.4349
LSM	0.0000	3.3257	0.1040	0.0892
GPM	0.0000	2.5011	0.0002	0.0500
LLSM	0.7034	0.0000	0.2423	3.3947
LSM	1.2331	0.0000	0.2532	3.5711
GPM	0.4172	0.0000	0.0005	3.8332
LLSM	0.3472	1.2663	0.0000	0.5493
LSM	0.5254	1.2971	0.0040	0.2304
GPM	0.0008	0.0011	0.0000	0.0001
LLSM	3.4249	1.4022	1.0470	0.0000
LSM	1.5424	2.0814	0.6309	0.0000
GPM	0.9992	**5.7466**	0.0004	0.0000

Table 3.3: Individual absolute deviations

$|a_{ij} - u_{ij}|$ (Example 3.1)

Method	Col. 1	Col. 2	Col. 3	Col. 4	Col. 5
LLSM	0.0000	0.0372	0.0564	0.0860	2.9722
LSM	0.0000	0.0152	0.0604	0.0407	3.8776
GPM	0.0000	0.0000	0.0007	0.0417	3.6674
LLSM	1.0959	0.0000	0.0856	0.0347	1.9444
LSM	0.6000	0.0000	0.1940	0.0934	0.5918
GPM	0.0000	0.0000	0.0000	0.0000	0.0045
LLSM	0.4384	0.0223	0.0000	0.0445	0.1944
LSM	0.6545	0.0537	0.0000	0.1054	0.8980
GPM	0.0000	0.0000	0.0000	0.0000	1.0032
LLSM	3.2603	0.0335	0.1497	0.0000	4.6111
LSM	1.9636	0.0854	0.3483	0.0000	1.7755
GPM	2.0000	0.0000	0.0000	0.0000	2.9955
LLSM	0.2932	0.0244	0.0075	0.0960	0.0000
LSM	0.6909	0.0100	0.0438	0.0524	0.0000
GPM	0.5504	0.0001	0.0501	0.7490	0.0000

3.6 Discussion

There are several important points which can be noted down from Table 3.1 through Table 3.4.

1. From Table 3.1 and 3.2, we note that the total absolute deviations due to GPM for both the examples are less than the corresponding values calculated by LLSM and LSM.

2. From Tables 3.3 and 3.4, we observe that the total number of satisfied and 'almost satisfied' cells in GPM are more than the corresponding numbers in LLSM and LSM. Therefore, it may be concluded that the weights obtained by GPM better fit the articulated responses in the matrices.

3. The inconsistent element can be easily identified by GPM (see cell a_{42} in Table 3.3). On this point elaborate discussion is available in Ignizio (1982, page 250-252).

4. SSE is the least in LSM, as expected.

5. From Table 3.1, we observe that the weights derived by various methods are significantly dispersed indicating presence of large amount of inconsistencies in the elements. The weights in Table 3.2 do not differ much due to minor inconsistency.

6. Dyer and Wendell (1985) and Holder (1990) have discussed the drawbacks of Saaty's (1/9-9) ratio-scale. Suppose, the object O_i is slightly more preferable to the object O_j, then according to Saaty's scale, the corresponding ratio is 2. But in any way, this does not mean that O_i is 2 times preferable than O_j, i.e., if any weight elicitation technique gives the weight of O_i and O_j in the exact ratio 2:1, then this will not match with Saaty's interpretation. There are many examples in AHP literature, where this anomaly occurs. If decision makers want to use GPM as a weight elicitation technique, then we should take the meaning of ratio-scale in the ordinary sense. The typical question, which is to be asked, is: "between two objects which one is more preferable and how many times (instead of how much more)?" By this way, DM may violate the limits of the (1/9-9) scale. Also he/she may be allowed to give any number between the limits of the scale. If the scale is interpreted in the foregoing manner, then GPM will give 'better' results than EM, LLSM or LSM. This fact is indicated by the more number of zeros in Table 3.4 corresponding to GPM.

3.7 Performance of Goal Programming Method on Various Criteria

The six potential criteria to evaluate performances of various weight determination techniques are:

(i) element preference reversal avoidance,

(ii) moderate rank row preference reversal minimization,

(iii) weak rank preference reversal minimization,

(iv) strong rank preference reversal minimization,

(v) row (column) permutation invariance, and

(vi) respond bound violation minimization and minimization of magnitudes of surrogate weight ratios.

The definitions of these criteria are given in the previous chapter. Experimentally, it is tested that ‘element preference reversal’ and ‘moderate rank preference reversal’ are unavoidable in GPM as in other methods, whereas GPM avoids ‘strong rank preference reversal’ always.

In Example 3.1, LSM fails to avoid weak rank preference reversal, whereas GPM preserves the same.

Jensen (1989) has proved that all additive error models (note that GPM belongs to additive error models) are row (column) permutation invariant. Although GPM fails to minimize the magnitudes of surrogate ratios, but it may give lesser number of respond bound violations than LSM (see Tables 3.3 & 3.4).

3.8 Concluding Remarks

There is a proliferation of methods for estimation of weights from Saaty’s pairwise comparison matrices. But none has emerged as the ‘best.’ The well-known least squares method is adopted to minimize the Euclidean distances. But in the weight extraction problem, decision maker desires to obtain that set of weights for which pairwise ratios are closest to the corresponding a_{ij} values in the comparison matrix. Naturally, goal programming can be utilized to achieve this ‘goal’. In this chapter, we have proposed goal programming method as an alternative ratio scaling technique and shown its superiority over LSM and LLSM with respect to a number of aspects of ratio-scale matrices. Empirically, it is observed that goal programming technique satisfies maximum number of elements in a comparison matrix. With respect to this criterion, GPM outperforms all other techniques.

Lastly, we emphasize the fact that, in general, there is no single weight elicitation technique which satisfies all potential concerns of the DM. The technique should be chosen in accordance with the concerns kept in DM’s mind.