Chapter 2

Determination of Weights From Pairwise Comparison Matrices in Analytic Hierarchy Process: A Fuzzy Programming Approach

2.1 Introduction

In real world, several decision making problems do incorporate uncertainties. These uncertainties can be broadly classified into two categories: probabilistic uncertainty and fuzzy uncertainty. Apart from these uncertainties, it is also recognized that most of the decision making problems involve multiple criteria and some of these criteria may be subjective or fuzzy in nature. The classical theory of probability is not sufficient to solve problems involving fuzzy uncertainty.

The Analytic Hierarchy Process (AHP) is a useful technique to deal with fuzzy criteria in solving discrete multiple criteria decision making (MCDM) problems (Saaty, 1978). Formation of pairwise comparison matrices and determination of relative weights of objects from them are the most important aspects of AHP. In the next section, we shall discuss various types of pairwise comparison matrices and their corresponding weight determination techniques.

2.2 Various Types of Comparison Matrices and Corresponding Weight Determination Techniques

To determine relative weights of n objects in the framework of AHP, one first constructs pairwise comparison matrices of the form:

$$\mathbf{A} = \begin{array}{c|cccc} & O_1 & O_2 & \dots & O_n \\ \hline O_1 & a_{11} & a_{12} & \dots & a_{1n} \\ O_2 & a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O_n & a_{n1} & a_{n2} & \dots & a_{nn} \end{array} \tag{2.1}$$

where $a_{ij} = w_i / w_j$, $i, j = 1, 2, ..., n$, $a_{ii} = 1$ and w_i's, $i = 1, 2, ..., n$, are the underlying relative weights of the n objects O_i. Here, different cases arise in articulating the preference ratios, a_{ij}'s, depending upon the decision making environment.

Case 1 *a_{ij}'s are crisp numbers such as 5 or 1/3 taken from the (1/9-9) scale. This is the usual practice in AHP.*

Case 2 *a_{ij} 's are fuzzy numbers such as (4, 5, 6) or (3, 4, 5, 6) with triangular or trapezoidal membership functions, respectively.*

Case 3 *a_{ij} 's are intervals of numbers such as [2, 5], [1/3, 4], or [1/4, ½]. The end points of the intervals are taken from the same (1/9-9) scale.*

Case 4 *a_{ij} 's are sets of points such as $a_{ij} = \{x \mid 3 \leq x\}$ or $a_{ij} = \{x \mid x \leq 7\}$.*

Case 5 *a_{ij} 's are approximate numbers such as ≈ 5 or $\approx 1/2$.*

As described in Chapter 1, there are mainly three methods to determine weights from comparison matrices in Case 1, namely eigenvector method (EM) (Saaty, 1977a), least squares method (LSM) (Jensen, 1984), and logarithmic least squares method (LLSM) (Crawford and Williams, 1985). For fuzzy matrices in Case 2, Van Laarhoven and Pedrycz (1983) used logarithmic least squares method to elicit weights. They adopted fuzzy numbers with triangular membership functions. Using fuzzy numbers with trapezoidal membership functions, Buckley (1985) used geometric mean method to elicit fuzzy weights. Arbel (1989) and Arbel and Vargas (1993) introduced Linear Programming (LP) approach in the AHP. They used LP method to extract weights from interval judgments as in Case 3, although the LP approach fails to find weights from inconsistent interval judgments. LP method can also be used to determine weights from comparison matrices where the elements a_{ij} are stated as in Case 4. But there does not exist any method for determination of weights from pairwise comparison matrices whose elements are appropriately articulated as in Case 5. In this chapter, a Fuzzy Programming method is presented to determine weights from appropriate comparison matrices.

2.3 Formulation of Fuzzy Programming as a Weight Determination Technique

To determine weights from appropriate pairwise comparison matrices, two different fuzzy programming methods are developed.

Method 1: Let us consider the relations

$$w_i / w_j \approx a_{ij}, \quad a_{ij} \geq 1 \quad \text{for all} \quad i, j \quad \text{and} \quad i \neq j \tag{2.2}$$

Now two cases may arise. In one case, the decision maker (DM) may be satisfied with 'small' deviations on both sides from a_{ij}. If this is the case, then the Relations (2.2) become

$$w_i / w_j = a_{ij} \pm p_{ij}, \quad a_{ij} \geq 1 \quad \text{for all} \quad i, j \quad \text{and} \quad i \neq j$$

where p_{ij} 's are subjectively chosen constants, called tolerance limits. In another case, DM may prefer small deviation only on one side from a_{ij}, i.e., in this case, either

$$w_i / w_j = a_{ij} + p_{ij} \qquad \text{or} \qquad w_i / w_j = a_{ij} - p_{ij}$$

Considering the former case, we have

$$(w_i - a_{ij} w_j)/(\pm p_{ij} w_j) = 1 \tag{2.3}$$

Each of the constraints of type (2.2) shall now be represented as a fuzzy set, for which the membership function of a_{ij} is

$$\mu_{ij}(w) = 1 - \frac{w_i - a_{ij} w_j}{p_{ij} w_j} \qquad \textit{(taking ‘+’ sign in (2.3))}$$

where

$$\mathbf{w} = (w_i, w_j)^T, \quad \textit{for all} \quad i, j \quad \textit{and} \quad i \neq j.$$

Therefore, $$\mu_{ij}(\mathbf{w}) = \begin{cases} 1 & \textit{if} \quad w_i - a_{ij} w_j = 0, \\ 0 < w_{ij} < 1 & \textit{if} \quad 0 < w_i - a_{ij} w_j < p_{ij} w_j, \\ 0 & \textit{if} \quad w_i - a_{ij} w_j \geq p_{ij} w_j. \end{cases}$$

For minus sign of $p_{ij}\ w_j$ in Equation (2.3), the signs of w_i and w_j will just be interchanged. The membership function $\mu_{ij}(w)$ can be interpreted as the degree to which the fuzzy equality $w_i / w_j \approx a_{ij}$ is satisfied.

Now let us define $\mathbf{A_{ij}} = (1 \quad -a_{ij})$ for $i = 1, 2, \ldots, n-1;\ j = i+1, \ldots, n$ (assuming $a_{ij} \geq 1$). Hence, $A_{ij} w = w_i - a_{ij} w_j$. Therefore, from the foregoing discussion we can write

$$\mu_{ij}(\mathbf{w}) = 1 - \frac{(\mathbf{A_{ij}} \mathbf{w}) - 0}{\pm p_{ij} w_j} \tag{2.4}$$

Following Bellman and Zadeh (1970), the membership function of the fuzzy set “decision” is given by

$$\mu_D(\mathbf{w}) = \min_{i,j} \mu_{ij}(\mathbf{w}) \tag{2.5}$$

Let us assume that the DM is interested for a decision not in a fuzzy set but a crisp optimal compromise solution. The objective is to find the maximizing solution which minimizes the deviations from the articulated values. Therefore, the solution is given by

$$\max_{\mathbf{w} > 0} \ \min_{i,j} \left[1 - \frac{(A_{ij} w) - 0}{\pm p_{ij} w_j} \right]$$

Introducing an augmented variable λ which is the minimum of all μ_{ij} ’s in (2.4), we arrive at the following non-linear programming problem:

Maximize λ

subject to
$$\lambda\, p_{ij} w_j + (\mathbf{A_{ij} w}) \le p_{ij} w_j$$
$$\lambda\, p_{ij} w_j - (\mathbf{A_{ij} w}) \le p_{ij} w_j \qquad i = 1, 2, \ldots, n-1; \quad j = i+1, \ldots, n$$
$$\sum_{i=1}^{n} w_i = 1$$
$$0 \le \lambda \le 1$$
$$w_i \ge 0, \quad i = 1, 2, \ldots, n$$

Remark 2.1: In the foregoing formulation, we considered only the upper triangular part of the comparison matrix. As stated in Section 2.2, the lower triangular part is just the reciprocals of the corresponding elements in the upper part. So, no additional information is provided by the elements in the lower triangular part. In addition to this, we have assumed that all the a_{ij}'s elements in Method 1 are greater than or equal to 1. If some a_{ij}'s elements in the upper triangular part are less than 1, then, for convenience of specifying p_{ij} values, the variables w_i and w_j may be interchanged after taking reciprocals of a_{ij}'s.

Method 2: Although subjective quantities p_{ij} can be chosen in Method 1 with ease, the decision maker has to solve a non-linear programming problem. In addition to this, for small values of p_{ij}, one may not get a positive value of λ.

The relation $w_i / w_j \approx a_{ij}$ may also be viewed as $w_i - a_{ij} w_j \approx 0$. Now the decision maker may accept the solution when $w_i - a_{ij} w_j = \pm p_{ij}, p_{ij} > 0$ being a very small quantity, preferably less than 0.10. Then, proceeding as in Method 1, in the present case, we have the following fuzzy linear programming problem:

Maximize λ

subject to
$$\lambda\, p_{ij} + (\mathbf{A_{ij} w}) \le p_{ij}$$
$$\lambda\, p_{ij} - (\mathbf{A_{ij} w}) \le p_{ij} \qquad i = 1, 2, \ldots, n-1; \quad j = i+1, \ldots, n$$
$$\sum_{i=1}^{n} w_i = 1$$
$$0 \le \lambda \le 1$$
$$w_i \ge 0, \quad i = 1, 2, \ldots, n$$

Remark 2.2: Knowing the p_{ij} value in Method 1 for some constraint, it is not possible to find the corresponding value in Method 2 for the same constraint, because of the variable w_j.

2.4 Numerical Examples

Example 2.1: Let us consider the following approximate pairwise comparison matrix:

	O_1	O_2	O_3	O_4	O_5
O_1	1	≈ 6	$*$	≈ 4	$\left.\begin{matrix}\approx 5\\ \approx 4\end{matrix}\right\}$
O_2		1	1/3	$1/\approx 2$	$1/\approx 2$
O_3			1	$\left.\begin{matrix}\approx 2\\ \approx 3\end{matrix}\right\}$	3
O_4				1	[1,2]
O_5					1

It is to be noted that no judgment is available for cell $a_{13.}$ For cells a_{15} and a_{34}, two judgments are available. Cells a_{23} and a_{35} have exact numbers. Interval judgment is considered for the element a_{45}.

The non-linear programming formulation of the foregoing weight determination problem is as follows:

$$
\begin{aligned}
&\text{Maximize:} \quad \lambda \\
&\text{subject to} \\
&1.25\lambda\, w_2 + w_1 - 6w_2 \le 1.25w_2 \\
&1.25\lambda\, w_2 - w_1 + 6w_2 \le 1.25w_2 \\
&1.25\lambda\, w_4 + w_1 - 4w_4 \le 1.25w_4 \\
&1.25\lambda\, w_4 - w_1 + 4w_4 \le 1.25w_4 \\
&1.25\lambda\, w_5 + w_1 - 4.47w_5 \le 1.25w_5 \\
&1.25\lambda\, w_5 - w_1 - 4.47w_5 \le 1.25w_5 \\
&3w_2 - w_3 = 0 \\
&1.25\lambda\, w_2 + w_4 - 2w_2 \le 1.25w_2 \\
&1.25\lambda\, w_2 - w_4 + 2w_2 \le 1.25w_2 \\
&1.25\lambda\, w_2 + w_5 - 2w_2 \le 1.25w_2 \\
&1.25\lambda\, w_2 - w_5 + 2w_2 \le 1.25w_2 \\
&1.25\lambda\, w_4 + w_3 - 2.45w_5 \le 1.25w_4 \\
&1.25\lambda\, w_4 - w_3 - 2.45w_5 \le 1.25w_4 \\
&w_3 - 3w_5 = 0 \\
&-w_4 + w_5 \le 0 \\
&w_4 - 2w_5 \le 0 \\
&w_1 + w_2 + w_3 + w_4 + w_5 = 1 \\
&\lambda, w_1, w_2, w_3, w_4, w_5 \ge 0
\end{aligned}
$$

From the above formulation (assuming all the tolerance limits p_{ij}'s for the approximate numbers as 1.25), we determine weights of the five objects by using the non-linear programming package GINO. The same weight determination problem has been formulated as a linear programming problem and solved by using the linear programming package LINDO. The two sets of weights are shown in Table 2.1.

Table 2.1: Relative weights of the five objects in Example 2.1.

Method	w_1	w_2	w_3	w_4	w_5	p_{ij}	λ
Method 1	0.4729	0.0846	0.2594	0.0945	0.0846	1.25	0.200
Method 2	0.4527	0.0827	0.2481	0.1338	0.0827	0.10	0.173

Example 2.2:

	O_1	O_2	O_3	O_4
O_1	1	1	7	4
O_2		1	6	3
O_3			1	1/4
O_4				1

In this example, we assume that all the preference strengths are articulated by exact numbers. The consistency ratio (C.R.) of the matrix is 0.088. Although this much inconsistency is allowed, but still the comparison matrix is inconsistent. So, it is impossible to satisfy all the constraints of type $a_{ij} = w_i / w_j$. Therefore, some deviations from the entries should be tolerated. Unlike the existing methods, in fuzzy programming method, DM has the advantage to specify the tolerance limits for violations of each (or some) of the constraints. The weights of the objects determined by Method 1 and Method 2 are shown in Table 2.2.

Table 2.2: Relative weights of the four objects in Example 2.2.

Method	w_1	w_2	w_3	w_4	p_{ij}	λ
Method 1	0.4326	0.3709	0.0543	0.1421	*	0.0778
Method 2	0.4358	0.3846	0.0512	0.1282	0.10	0.2307

* $p_{12} = 0.5$, $p_{13} = 1.5$, others = 1.2

For simplicity, same tolerance limits are assumed for all the constraints in Example 2.1 and most of the constraints in Example 2.2. Actually, it depends on how much the DM can deviate from his/her own preference strengths. In case the DM does not want to go beyond some specified tolerance limits, but still the value of λ is zero, then he/she must decrease the inconsistencies among the entries of the matrix. (The algorithm presented in the Chapter 5 may be used to decrease inconsistencies).

Remark 2.3: The closer the value of λ to 1, the more is the degree of satisfaction for all the constraints. To get higher value of λ from some lower positive value, DM has to increase the lengths of tolerance intervals. But the DM may accept the solution corresponding to lower positive value of λ (as in Table 2.2) satisfying all his/her articulated preference strengths.

2.5 Performance of Fuzzy Programming Method (FPM) on Various Criteria

Jensen (1989) has identified seven potential criteria to evaluate performance of various weight determination techniques. We will discuss them briefly to evaluate performance of FPM.

2.5.1 Element Preference Reversal (EPR)

Consider any element $a_{ij} > 1 \ (or \ < 1)$ in a comparison matrix. If for the estimated weights w_i and $w_j, u_{ij} = w_i / w_j < 1 \ (or \ > 1)$, then we say that *element preference reversal* has taken place.

For a considerably inconsistent matrix, EPR can never be avoided by any weight estimation technique. But Jensen (1989) has commented that eigenvector method minimizes the cardinal magnitudes of such reversals. After the development of FPM his statement does not appear to be true. Let us consider the same example adopted by Jensen (1989, page 6):

	O_1	O_2	O_3		O_1	O_2	O_3		O_1	O_2	O_3
O_1	1	$\boxed{2}$	1	O_1	1	$\boxed{0.76}$	2.62	O_1	1	$\boxed{1.11}$	5.25
O_2		1	9	O_2		1	3.43	O_2		1	4.75
O_3			1	O_3			1	O_3			1
Comparison matrix				Matrix obtained from EM solution				Matrix obtained from FPM solution (Method 2)			

It is to be noted that EPR has been transpired in EM for a_{12}. But FPM not only preserves the preference, but in this case $w_1 / w_2 = 1.11 > 0.76$.

Actually, EM is designed to preserve ordinal ranking, and in this respect, according to Jensen (1989), it outperforms other existing methods. Let us consider the following highly inconsistent matrix with C.R. = 0.896 adopted by Saaty and Vargas (1984b):

	O_1	O_2	O_3	O_4
O_1	1	4	½	1/5
O_2		1	1/3	4
O_3			1	½
O_4				1

For this example, the weight vectors obtained by EM and FPM are respectively, (0.214, 0.245, 0.242, 0.299) and (0.340, 0.164, 0.376, 0.120). Even for this highly inconsistent matrix, we observe that the number of EPRs is the same in EM and FPM (both = 2). Based upon the results of the previous two examples, it can be said that FPM is better in the sense of minimization of EPRs.

2.5.2 Moderate Rank Preference Reversal (MRPR)

Let us assume, for any pairwise comparison matrix (PCM), $a_{ij} \geq 1 \; \forall \; j \; and \; a_{ij} > 1$ for at least one j. Now if the estimated weights $w_i \; and \; w_j$ are such that $w_j > w_i$ for some j $\neq i$, then this indicates *moderate rank preference reversal.*

Let us consider the matrix (Jensen, 1989, page 7):

	O_1	O_2	O_3	O_4	O_5
O_1	1	2	2	2	2
O_2		1	9	9	9
O_3			1	1	1
O_4				1	1
O_5					1

It is to be noted that a_{1j} >1 for j = 2, 3, 4, 5, i.e., w_1 should be greater than w_j , j = 2, 3, 4, 5. The weights of the objects O_i, $i = 1, 2, 3, 4, 5,$ calculated by various methods, are shown in Table 2.3.

Table 2.3: Weights of the five objects calculated by various methods

Methods	w_1	w_2	w_3	w_4	w_5
EM	0.300	0.472	0.076	0.076	0.076
LLSM	0.261	0.487	0.084	0.084	0.084
LSM	0.200	0.596	0.068	0.068	0.068
FPM	0.428	0.357	0.071	0.071	0.071

From Table 2.3, we note that $w_2 > w_1$ for the first three existing methods. That's why Jensen (1989) concluded that *"moderate rank preference reversals are frequently impossible to avoid…"* It is worth noting that only FPM avoids the MRPR. So, MRPR can be avoided and this can only be achieved by FPM.

2.5.3 Weak Rank Preference Reversal (WRPR)

Let us assume that, for any PCM, some $a_{ik} \geq a_{jk}$ but other $a_{ik} \leq a_{jk}$, for some particular i and j and k = 1, 2, …, n. In this case, we say that the ith row is weakly dominant over the jth row. Now, if the estimated weights w_i and w_j are such that $w_j > w_i$, then we say that *weak rank preference reversal* has occurred.

Like EM and unlike LSM, FPM avoids WRPR. This may be verified by taking the same example adopted by Jensen (1989, page 8).

2.5.4 Strong Rank Preference Reversal

If for any PCM, $a_{ik} \geq a_{jk}$ for some particular i and j and all k, then we say that the ith row has strong preference over the jth row.

It is fairly easy to verify that all the existing methods including FPM avoid strong rank preference reversal.

2.5.5 Row (Column) Permutation Invariance

A weight determination method is called *row (column) permutation invariant,* if the weight (determined by the concerned method) of a particular object O_i, i = 1, 2, …, n, remains invariant under any permutation of the objects (Jensen, 1989). All the additive error models including the FPM are row (column) permutation invariant.

2.5.6 Inverse Reciprocity Property

The element a_{ij} denotes the extent to which object O_i is better than object O_j. Therefore, the elements $b_{ij} = 1/a_{ij} = a_{ji}$ will denote the extent to which object O_i is worse than object O_j. Now the weights associated with B = [b_{ij}] should be the reciprocals of the corresponding weights associated with A = [a_{ij}]. This is called *inverse reciprocity property.* It may be recalled that, in our fuzzy programming method, we have considered only the upper triangular part of PCM and all the technological coefficients in the formulation of fuzzy programming method in Section 2.3 are considered greater than or equal to 1 just for convenience to provide the tolerance limits p_{ij}'s. Now in the dual case, difficulties arise providing p_{ij} values because here all the technological coefficients are less than 1. Even if DM provides p_{ij} values, those may not match with the corresponding p_{ij} values in the primal case. So FPM, in general, fails to preserve inverse reciprocity property.

2.5.6 Response Bound Violation (RBV)

As mentioned in Section 2.2, the responses are provided by taking discrete points from Saaty's (1/9-9) scale. After estimating weights and taking their pairwise ratios, it may be observed that some ratios violate the upper bound 9 (or lower bound 1/9) of the scale. This phenomenon is known as *response bound violation.*

To discuss RBV, Jensen (1989, page 14) has considered the wealth nation comparison matrix originally adopted by Saaty and Khouja (1976). For this matrix, the ratios of w_1 and w_3 determined by EM, LSM, and FPM are, respectively, 20.43, 10.70, and 11.68 where original response is $a_{13} = 9$. So EM suffers badly on RBV, whereas LSM not only gives minimum number of RBVs, but also minimizes the magnitudes of surrogate ratios. Our FPM remains close to LSM with respect to this criterion.

2.6 Advantages and Disadvantages of Fuzzy Programming Method as a Weight Determination Technique

i) While eliciting preference responses, DM may not be certain about all the comparisons. One should not force DM to give his/her preference strength as a crisp or exact number in an uncertain case, rather he/she can state that by means of an approximate number. FPM can be conveniently used to determine the weights of alternatives from such approximate matrices.

ii) In the AHP framework, all the constraints are of the type

$$w_i - a_{ij} w_j = 0 \quad \forall \;\; i, \;\; j \, .$$

In the language of Zimmermann (1985, page 221)
"The ≤ sign might not be meant in the strictly mathematical sense, but smaller violations might well be acceptable. This can happen if the constraints represent sensory requirements (taste, color, smell, etc.) which cannot adequately be approximated by a crisp constraint."

In AHP, the foregoing '≤' sign should be read as '=' sign. No existing method except FPM considers the aforementioned flexibility regarding constraint satisfaction.

iii) It is already mentioned that, in fuzzy situations, DM may state his/her preferences either by fuzzy numbers or by approximate numbers. If he/she uses fuzzy numbers, then the overall scores of the alternatives will also be fuzzy. So, one needs the ranking of these fuzzy numbers, which itself is a complex problem (Bortolan and Degani, 1985). Also clearcut ranking is not possible for fuzzy scores. In fuzzy ranking, some alternative is superior to some extent, beyond which that alternative ceases to be superior. By means of FPM, however, we obtain clearcut ranking of all the alternatives.

iv) Several types of constraints can be considered in fuzzy programming approach, viz., rigid constraints, flexible constraints, constraints obtained from interval judgments, etc.

v) Missing judgments can obviously be handled by fuzzy programming method. Multiple judgments can be aggregated as an interval judgment; alternatively Saaty's (1989) geometric mean procedure can be used.

vi) Sensitivity analysis can easily be done in FPM.

vii) As mentioned in Section 2.5, FPM is the only method by which one can avoid 'moderate rank preference reversal'.

Despite the aforementioned advantages, FPM has some disadvantages too.

i) Specification of tolerance limits (p_{ij} values) may be extra burden to the DM in addition to the task of filling up the requisite number of comparison matrices.

ii) DM may not get positive value of λ or rather feasible solution of the problem, even after specification of p_{ij} values. To get positive value of λ, DM may need to increase of the length of tolerance intervals further, which he/she may not want to. In this case, FPM fails to determine weights.

2.7 Concluding Remarks

While comparing two alternatives with respect to some highly subjective criteria such as attractiveness, taste, comfort, etc., DM may not be fully confident about the choice of a discrete point from the (1/9-9) scale. In fact, in this case, preference strengths cannot be adequately expressed by means of exact numbers; rather, it may be easier for DM to state his/her preference strengths by means of approximate numbers. Of course, if DM feels certain about some particular comparison, then he/she should state that preference strength by exact number. Even after stating preference ratios by exact numbers, in some cases, DM may be satisfied with small deviations from their stated ratios. In such cases, we have shown how fuzzy programming technique can be used to determine weights of objects from comparison matrices. One may choose either Method 1 or Method 2 according to his/her convenience. The weight determination problem formulated in Method 1 and 2 can also be solved by linear programming method (Arbel, 1989), because after specifying the tolerance limits every entry becomes an interval judgment provided the converted interval matrix is consistent. However, the objectives of FPM and LP methods are just opposite to each other; FPM tries to minimize the deviations from the articulated ratios, whereas, LP method finds out solutions ranging over the whole interval.